A note on the knapsack problem with special ordered sets

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RESUME :

Nous montrons que le problème du sac à dos avec des bornes supérieures généralisées et des coefficients de signe arbitraire est équivalent à un problème de ce type qui n'a que des coefficients positifs ; ce dernier problème est dit standard. Nous démontrons deux propositions qui donnent un algorithme pour le programme linéaire (relaxé) associé au problème standard. Cet algorithme est une généralisation naturelle de l'algorithme de Dantzig pour le problème du sac à dos sans bornes supérieures généralisées. Nous dérivons aussi plusieurs propriétés de l'enveloppe convexe des solutions zéro-un pour cette classe de problèmes.

ABSTRACT :

The knapsack problem with special ordered set and arbitrarily signed coefficients is shown to be equivalent to a standard problem of the same type but having all coefficients positive. Two propositions are proven which define an algorithm for the linear programming relaxation of the standard problem that is a natural generalization of the Dantzig solution to the problem without special ordered sets. Several properties of the convex hull of the associated zero-one polytope are derived.

Key words : knapsack problem, special ordered sets, GUB, algorithms, facets.

INTRODUCTION

Consider the following problem

$$\max \sum_{i\in I} \sum_{j\in K_i} c_j x_j$$

subject to

$$\sum_{i\in I} \sum_{i\in K_i} a_j x_j \le a_o$$

(KPSOS)

$$\sum_{j\in K_i} x_j \le 1 \qquad \text{for all } i\in I$$

$$x_j \ge 0 \qquad \text{for all } j\in K_i,\ i\in I$$

where the K_i satisfy $|K_i| \geq 1$ for all $i\in I$, $K_i \cap K_j = \phi$ holds for all $i \neq j$ and the data c_j and a_j are arbitrarily signed real numbers. This problem has received much attention in recent literature, see e.g. Zemel [6] for a survey of the literature. In this note we first show that (KPSOS) can always be brought into a "standard" form with positive data and prove two propositions which define an algorithm for (KPSOS) that is a natural extension of Dantzig's solution [2] to (KPSOS) when $|K_i| = 1$ for all $i\in I$. Note that arbitrary upper bounds u_i on the special ordered sets can be dealt with by scaling. We then consider the zero-one-variables version of (KPSOS) and derive several properties of the facets of the associated convex hull of zero-one solutions to (KPSOS).

1. The Algorithm

Note that any special ordered set constraint of the form $\sum_{j\in K_i} x_j = 1$ can be brought to the inequality form by eliminating one of the variables in such set. Next we show that (KPSOS) can always be brought to a standard form with all data positive. Define

$$a_{j_i} = \min\{a_j \mid j \in K_i\} \quad \text{for } i \in I \tag{1.1}$$

and let $H = \{i \in I \mid a_{j_i} < 0\}$. For all $i \in H$ we set

$$a_j' = \begin{cases} a_j - a_{j_i} & \text{for all } j \in K_i - \{j_i\} \\ -a_{j_i} & \text{for } j = j_i \end{cases}$$

$$c_j' = \begin{cases} c_j - c_{j_i} & \text{for all } j \in K_i - \{j_i\} \\ -c_{j_i} & \text{for } j = j_i \end{cases} \tag{1.2}$$

$$a_o' = a_o - \sum_{i \in H} a_{j_i}.$$

Let $K = \bigcup_{i \in H} K_i$ and define $a_j' = a_j'$, $c_j' = c_j$ for all $j \notin K$. Consider the problem

$$\max \sum_{i \in I} \sum_{i \in K_i} c_j' z_j$$

subject to

$$\sum_{i \in I} \sum_{j \in K_i} a_j' z_j \leq a_o'$$

(KPSOS*)

$$\sum_{j \in K_i} z_j \leq 1 \qquad \text{for all } i \in I$$

$$z_j \geq 0 \qquad \text{for all } j \in K_i,\ i \in I.$$

Problem (KPSOS*) satisfies $a'_j \geq 0$ for all j and the variable substitution

$$
(1.3) \qquad \begin{array}{lll}
x_j = z_j & \text{for } j \notin K & \\
x_j = z_j & \text{for } j \in K_i - \{j_i\} & \multirow{2}{*}{$\left\{ \text{for all } i \in H \right.$} \\
x_j = 1 - \sum\limits_{j \in K_i} z_j & \text{for } j = j_i &
\end{array}
$$

provides a one-to-one mapping between feasible solutions to (KPSOS) and (KPSOS*), respectively, which preserves - up to the constant term $\sum\limits_{i \in H} c_{j_i}$ – the value of the objective function.

If $c_j' \le 0$ and $a_j' \ge 0$ in (KPSOS*) we set $z_j = 0$; if $c_j' > 0$ and $a_j' = 0$ for some $j \in K_i$ and some $i \in I$, it follows that every optimum solution to (KPSOS*) satisfies $\sum\limits_{j \in K_i} z_j = 1$ and letting

$$
(1.4) \qquad c'_{j_*} = \max \{c'_j \mid a'_j = 0,\ j \in K_i\}
$$

we can eliminate variable z_{j_*} from the problem. The eliminated variable is recorded and its value determined at the end of the calculation. After this reduction (possibly used repeatedly) we thus obtain the linear knapsack problem with special ordered sets in the following standard form:

$$\max \sum_{i \in L} \sum_{j \in S_i} d_j z_j$$

subject to

(KPSOSX)

$$\sum_{i \in L} \sum_{j \in S_i} g_j z_j \leq g_o$$

$$\sum_{j \in S_i} z_j \leq 1 \qquad \text{for all } i \in L$$

$$z_j \geq 0 \qquad \text{for all } j \in S_i,\ i \in L$$

where $d_j > 0$, $g_j > 0$, $g_o > 0$ for all j and $|S_i| \geq 1$ for all $i \in L$. Let for $i \in L$

$$\frac{d_{j_i}}{g_{j_i}} = \max \{\frac{d_j}{g_j} \mid j \in S_i\} \tag{1.5}$$

and suppose that the special ordered sets have been ordered such that

$$\frac{d_{j_1}}{g_{j_1}} \geq \frac{d_{j_2}}{g_{j_2}} \geq \dots \geq \frac{d_{j_h}}{g_{j_h}} \tag{1.6}$$

where $h = |L|$.

Proposition 1.1: There exists an optimal solution z^* to (KPSOSX) such that $\sum_{k \in S_1} z^*_k = 1$ or an optimal solution is given by $z^*_{j_1} = g_o/g_{j_1}$, $z^*_j = 0$ for all $j \neq j_1$.

Proof: Suppose that $\sum_{k \in S_1} z_k < 1$ holds in every optimal solution to (KPSOSX). Let z be any optimal solution and $z_j > 0$ for some $j \neq j_1$. Define z^* by $z^*_{j_1} = z_{j_1} + \epsilon$, $z^*_j = z_j - \epsilon g_{j_1}/g_j$,

$z^*_k = z_k$ for all $k \neq j_1$, j where

$$(1.7) \qquad \epsilon = \min \{1 - \sum_{k \in S_1} z_k,\ z_j\, g_j/g_{j_1}\}.$$

Clearly $\epsilon > 0$ and z^* is a feasible solution to (KPSOSX). We then get

$$(1.8) \qquad \sum_{\text{all } k} d_k\, z^*_k - \sum_{\text{all } k} d_k\, z_k = d_{j_1}\epsilon - d_j \epsilon g_{j_1}/g_j = \left(\frac{d_{j_1}}{g_{j_1}} - \frac{d_j}{g_j}\right) g_{j_1}\epsilon \geq 0$$

and thus z^* is optimal as well. If $\epsilon = 1 - \sum_{k \in S_1} z_k$ holds, then $\sum_{k \in S_1} z_k^* = 1$. Else we can iterate unless $z^*_{j_1}$ is the only positive variable and $z^*_{j_1} = g_o/g_{j_1}$ holds since $\sum_{k \in S_1} z_k < 1$ holds in every optimal solution. The proposition follows.

<u>Proposition 1.2</u>: <u>If</u> $g_{j_1} \geq g_o$, <u>then</u> $z_{j_1} = g_o/g_{j_1}$, $z_k = 0$ <u>for all</u> $k \neq j_1$ <u>is an optimal solution to (KPSOSX)</u>.

Proof: If $g_{j_1} \geq g_o$ holds, the solution of proposition 1.2 is basic and a feasible basis is obtained by making all slack variables of the special ordered sets basic. Denoting s_o the slack of the knapsack constraint we compute

$$\sum_{i \in L} \sum_{j \in S_i} d_j\, z_j = d_{j_1} \frac{g_o}{g_{j_i}} + \sum_{j \neq j_1} \left(d_j - g_j \frac{d_{j_1}}{g_{j_1}}\right) z_j - \frac{d_{j_1}}{g_{j_1}} s_o$$

and thus the reduced cost of all nonbasic variables are nonnegative, establishing optimality of the proposed solution. The proposition follows.

Propositions 1.1 and 1.2 define an algorithm for the linear knapsack problem with special ordered sets which is a natural generalization of Dantzig's method for the linear knapsack problem with variables having upper bounds. Let $j_1, j_2, \ldots, j_k$ be defined as in (1.6). If $g_{j_1} \geq g_o$ holds, we are done by proposition 1.2. Else $g_{j_1} < g_o$ holds, thus we know by proposition 1.1 that

$$\sum_{j \in S_1} z_j = 1 \tag{1.9}$$

holds at the optimum. If $|S_1| = 1$ holds, we fix the corresponding variable, record the change of g_o and iterate. Else we eliminate from S_1 the variable j_* defined by

$$d_{j_*} = \max \{d_j \mid g_j = g^*\} \text{ where } g^* = \min \{g_j \mid j \in S_1\}. \tag{1.10}$$

We replace S_1 by the set

$$S_1^* = \{j \in S_1 \mid d_j > d_{j_*}\}, \tag{1.11}$$

replace d_j by $d_j - d_{j_*}$, g_j by $g_j - g_{j_*}$ for all $j \in S_1^*$ and replace g_o by $g_o - g_{j_*}$. If $S_1^* = \phi$ holds, we fix z_{j_*} at value 1 and iterate. If $S_1^* \neq \phi$ holds, we record the eliminated variable and compute

$$\frac{d_{j_+}}{g_{j_+}} = \max \left\{ \frac{d_j}{g_j} \mid j \in S_1^* \right\}. \tag{1.2}$$

We merge the special ordered set S_1^* into its proper place according to the value of this ratio among the remaining (decreasingly ordered) ratios to be worked at and iterate.

In the following iterations it may happen that the eliminated variable is set to zero if S_1^* is used again for elimination or that the eliminated variable is set to equal to $1 - g_o/g_{j_i}$ when the special ordered set corresponding to the current index i(i.e. in the redefined ordering) at which the algorithm stops corresponds to the original set S_1. In any case, it may be that a previously eliminated variable may have to be reset at such fractional value if the current index i corresponds to some other special ordered set which was used previously for elimination and the algorithm stops with at most two variables at fractional values.

2. PROPERTIES OF FACETS FOR THE ZERO-ONE POLYTOPE

We consider again the general knapsack problem with special ordered sets and require that all variables be zero-one variables. As we are interested in the convex hull of zero-one solution we dispense with the objective function and consider the constraint set

(C1)
$$\sum_{i \in I} \sum_{j \in K_i} a_j x_j \le a_o$$

$$\sum_{j \in K_i} x_j \le 1 \qquad \text{for all } i \in I$$

$$x_j = 0 \text{ or } 1 \qquad \text{for all } j \in_i,\ i \in I.$$

Using the variable substitution (1.3) we bring (C1) into the equivalent form

$$\sum_{i \in I} \sum_{j \in K_i} \alpha_j z_j \le \alpha_o$$

(C2)

$$\sum_{j \in K_i} z_j \le 1 \qquad \text{for all } i \in I$$

$$z_j = 0 \text{ or } 1 \qquad \text{for all } j \in K_i,\ i \in I.$$

where the α_j are non-negative and correspond to the a'_j of (1.2).

Proposition 2.1: Let $\pi z \le \pi_o$ be a facet (a valid inequality) of the convex hull of (C2), then

(2.1) $$\sum_{i \in H} (-\pi_{j_i}) x_{j_i} + \sum_{i \in H} \sum_{j \in K_i - \{j_i\}} (\pi_j - \pi_{j_i}) x_j + \sum_{j \in K} \pi_j x_j \le \pi_o - \sum_{i \in H} \pi_{j_i}$$

is a facet of (a valid inequality) of the convex hull of (C1). Conversely, let $\mu x \le \mu_o$ be a facet (a valid inequality) of the convex hull of (C1), then

(2.2) $$\sum_{i \in H} (-\mu_{j_i}) z_{j_i} + \sum_{i \in H} \sum_{j \in K_i - \{j_i\}} (\mu_j - \mu_{j_i}) z_j + \sum_{j \in K} \mu_j z_j \le \mu_o - \sum_{i \in H} \mu_{j_i}$$

is a facet (a valid inequality) of the convex hull of (C2), where the notation of Section 1 is used.

Proof: The variable substitution (1.3) can be written as $x = f + Az$ where f is a zero-one vector and A is a nonsingular matrix. On the other hand one verifies that $z = f + Ax$ holds as well. Validity of the respective inequalities follows from the fact that the variable substitution provides a one-to-one mapping of feasible solutions. The facet defining property of the respective inequalities follows from the non-singularity of A and thus the proposition follows.

We assume now without restriction of generality that $\alpha_j \leq \alpha_o$ holds for all j. It follows that the inequalities

$$(2.3)\qquad \begin{aligned} \sum_{j \in K_i} z_j &\leq 1 \qquad && \text{for all } i \in I \\ -z_j &\leq 0 \qquad && \text{for all } j \in K_i,\ i \in I \end{aligned}$$

define facets of the convex hull of (C2). We call those facets the trivial facets. Due the non-negativity of the constraint system (C2) we know e.g. from [3] that every facet $\pi z \leq \pi_o$ of (C2) which is not a non-negativity condition satisfies $\pi \geq 0$ and $\pi_o > 0$.

Proposition 2.2: If $\alpha_j = 0$ ~~for myself~~ for some $j \in K_i$, $i \in I$, then $\pi_j = 0$ in every nontrivial facet $\pi z \leq \pi_o$ of the convex hull of (C2).

Proof: Suppose the proposition is false and let $\pi z \leq \pi_o$ be a facet which has $\pi_j > 0$ for some $j \in K_i$, $i \in I$, with $\alpha_j = 0$. Since $\pi z \leq \pi_o$ is nontrivial there exists a feasible solution z^* to (C2) such that $\pi z^* = \pi_o$ and $\sum_{j \in K_i} z_j^* = 0$, since otherwise we have $\pi z = \sum_{j \in K_i} z_j$ and $\pi_o = 1$. But then z^{**} defined by $z^{**}_k = z^*_k$ for all $k \neq j$ and $z^{**}_j = 1$ is a feasible solution to (C2) satisfying $\pi z^{**} = \pi_o + \pi_j > \pi_o$, contradiction.

Consequently, we can purge all zero α_j in (C2) and assume that $\alpha_o \geq \alpha_j > 0$ holds for all $j \in K_i$ and all $i \in I$.

Assume next that the K_i are indexed such that

$$K_i = \{j_i, j_i + 1, \ldots, j_i + t_i\}$$

(2.4)

$$\alpha_{j_i} \leq \alpha_{j_i+1} \leq \ldots \leq \alpha_{j_i+t_i}$$

where $t_i \geq 0$ holds.

<u>Proposition 2.3:</u> <u>If $\pi z \leq \pi_o$ is a nontrivial facet of the convex hull of (C2) and for some $i\epsilon I$ and</u> $j\epsilon K_i$ <u>we have</u> $\pi_j > 0$, then

(2.5) $$\pi_k \geq \pi_j \quad \text{for all } k\epsilon K_i, \ k > j$$

<u>holds</u> .

Proof: Suppose the proposition is false. Since $\pi z \leq \pi_o$ is nontrivial, there exists a feasible solution z^* to (C2) with $\pi z^* = \pi_o$ and $z^*_k = 1$ since otherwise $\pi z = -z_k$ and $\pi_o = 0$ holds. Since $\alpha_j \leq \alpha_k$ it follows that z^{**} defined by $z^{**}_\ell = z^*_\ell$ for all $\ell \neq k,j$ and $z^{**}_k = 0$, $z^{**}_j = 1$ is feasible for (C2) and that $\pi z^{**} = \pi_o + \pi_j - \pi_k > \pi_o$ holds, a contradiction.

To state the next proposition we need the following auxiliary problem: For $k\epsilon I$ and $\ell\epsilon K_k$ let

$$\zeta_\ell = \max \sum_{\substack{i\epsilon I \\ i\neq k}}^{\alpha_j} \sum_{j\epsilon K_i} \alpha_j z_j$$

subject to

$$\sum_{\substack{i\epsilon I \\ i\neq k}} \sum_{j\epsilon K_i} \alpha_j z_j \leq \alpha_o - \alpha_\ell$$

(AUXKT)

$$\sum_{j\epsilon K_i} z_j \leq 1 \qquad \text{for all } i\epsilon I, \ i\neq k$$

$$z_j = 0 \text{ or } 1 \qquad \text{for all } j$$

<u>Proposition 2.4:</u> <u>If</u> $\zeta_j = \zeta_\ell$ <u>for</u> $\ell \neq j$, $\ell \in K_k$, $j \in K_k$ <u>and some</u> $k \in I$, <u>then</u> $\pi_j = \pi_\ell$ <u>for every nontrivial facet</u> $\pi z \leq \pi_o$ <u>of the convex hull of (C2).</u>

Proof: Suppose the proposition is false and let $\pi z \leq \pi_o$ be a facet with $\pi_j \neq \pi_\ell$. Without restriction of generality let $\pi_j < \pi_\ell$. Since $\pi z \leq \pi_o$ is a nontrivial facet of the convex hull of (C2) there exists a feasible zero-one vector z with $z_j = 1$ and $\pi z = z_o$. Define z^* by $z^*_\ell = 1$, $z^*_j = 0$, $z^*_h = z_h$ for all $h \neq \ell, j$. Then

$$\sum_{i \in K} \sum_{h \in K_i} \alpha_h z^*_h \leq \alpha_\ell + \zeta_j = \alpha_\ell + \zeta_\ell \leq \alpha_o$$

holds and thus z^* is feasible. But $\pi z^* = \pi_o + \pi_\ell - \pi_j > \pi_o$ holds, contradiction. Thus the proposition follows.

The assumption of proposition 2.4 is satisfied e.g. if $\alpha_j = \alpha_\ell$ holds. It follows that we can purge all but one of the variables in any special ordered set which have identical coefficients when we are interested in finding a minimal linear constraint set for the convex hull of (C2). We initially conjectured a stronger property, namely that all facets of the convex hull of (C2) could be obtained by "lifting" facets from associated knapsack inequalities with all special ordered sets consisting of singletons. While the conjecture is true for facets derived from minimal covers, see [3], [4], and from (1,k) configurations, see [5], the following example shows that the conjecture is false in general.

<u>Example 2.5</u>: Consider the inequalities

$$\begin{aligned} 5x_1 + 3x_2 + 3x_3 + x_4 + 2x_5 &\leq 7 \\ x_1 + x_2 \qquad\qquad\qquad &\leq 1 \end{aligned}$$

$$0 \leq x_i \leq 1 \text{ and integer, } i = 1,\ldots,5.$$

The inequality $2x_1 + x_2 + x_3 + x_4 + x_5 \leq 3$ defines a facet of the corresponding convex hull none of whose projections into a lower-dimensional space, however, defines a facet for the associated lower-dimensional zero-one problem.

The algorithm for the linear knapsack problem with special ordered sets as well as the properties of facets of the convex hull of (C2) are currently used in a cutting-plane-based approach to large-scale zero-one programming problems [1].

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